

CYCLICITY IN RANK-ONE PERTURBATION PROBLEMS

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ABSTRACT. The property of cyclicity of a linear operator, or equivalently the property of simplicity of its spectrum, is an important spectral characteristic that appears in many problems of functional analysis and applications to mathematical physics. In this paper we study cyclicity in the context of rank-one perturbation problems for self-adjoint and unitary operators. We show that for a fixed non-zero vector the property of being a cyclic vector is not rare, in the sense that for any family of rank-one perturbations of self-adjoint or unitary operators acting on the space, that vector will be cyclic for every operator from the family, with a possible exception of a small set with respect to the parameter. We discuss applications of our results to Anderson-type Hamiltonians.

1. INTRODUCTION

Consider a self-adjoint operator T on a separable Hilbert space \mathcal{H} . A vector $\varphi \in \mathcal{H}$ is called cyclic for an operator T , if

$$\mathcal{H} = \text{clos span}\{(T - \lambda \mathbf{I})^{-1}\varphi : \lambda \in \mathbb{C} \setminus \mathbb{R}\}.$$

An operator T is called cyclic, if there exists a cyclic vector. For a bounded operator T , an equivalent definition is that

$$\mathcal{H} = \text{clos span}\{T^n \varphi : n \in \mathbb{N} \cup \{0\}\},$$

i.e., the span of the orbit of φ under T is dense in the Hilbert space. Cyclicity of an operator is equivalent to the property that the operator has simple spectrum. The property of simplicity of the spectrum often appears in problems originating from physics.

In this note we study cyclicity in the context of rank-one perturbation problems for self-adjoint and unitary operators. If A is a self-adjoint operator and φ is its cyclic vector one can consider the family of rank-one perturbations

$$(1.1) \quad A_\alpha = A + \alpha(\cdot, \varphi)\varphi, \quad \text{for } \alpha \in \mathbb{R}.$$

Similar families can be defined for unitary operators, see sections 2.2 and 2.3 for the definitions. We show that the property of cyclicity for a fixed non-zero vector in the Hilbert space is not a rare event, in the sense that for any family of cyclic rank-one perturbations a fixed vector is cyclic for *all* operators in the family with an exception of some small sets of parameters.

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While similar statements, showing that any fixed non-zero vector is cyclic for almost every rank-one perturbation of any self-adjoint operator, exist in the literature (see [8, 9]), we attempt to make these statements more precise. In the rank-one setting we improve the existing results from "almost every" to "all but countably many" or even "all but possibly one" in some cases.

In particular, in section 3 we prove that an arbitrary non-zero vector φ in a Hilbert space is cyclic for all but countably many operators $(A_\alpha)_{ac}$ for any family of self-adjoint (or unitary) rank-one perturbations A_α . An arbitrary non-zero vector is also cyclic for almost all operators $(A_\alpha)_s$ for any such family. Here $(A_\alpha)_{ac}$ and $(A_\alpha)_s$ denote the absolutely continuous and singular parts of the operators A_α respectively, see section 2.1 for the definitions. This result strengthens a theorem by Jakšić and Last [8, 9].

In section 4, we show that if a vector belongs to a certain natural class of vectors, associated with the family A_α , then we have cyclicity for all except possibly one operator A_α .

In subsection 3.4 we discuss applications of our results to Anderson-type Hamiltonians and deduce some of the results from [8, 9] and [21].

The theory of rank-one perturbations of self-adjoint and unitary operators, and its applications to Anderson-type models became an active area of research over the last 20 years. The interest to this part of perturbation theory is caused, to a large degree, by connections to the famous problem of Anderson localization.

In 1958 P. W. Anderson (see [1]) suggested that sufficiently large impurities in a semiconductor could lead to spatial localization of electrons, called Anderson localization. Although most physicists consider the problem solved, many mathematical questions with striking physical relevance remain open. The field has grown into a rich mathematical theory (see e.g. [6, 7, 11] for different Anderson models and [5, 13] for refined notions of Anderson localization).

While the property of localization for a random Anderson-type operator has many different definitions, one of the "weaker" definitions of localization is equivalent to the property that the spectrum of the operator is almost-surely singular. It is well-known that, if an Anderson-type Hamiltonian is almost-surely singular, then it is almost-surely cyclic. Equivalently, if such an operator is not cyclic with positive probability, then it is delocalized.

The study of spectral behavior under rank-one perturbations proves to be one of the main tools in spectral analysis of Anderson-type models, in particular in problems concerning cyclicity, see for instance [8, 9, 21]. This connection served as one of the motivating factors for the current paper.

2. PRELIMINARIES

2.1. Cyclicity for normal operators. Recall that an operator in a separable Hilbert space is called normal if $T^*T = TT^*$. By the spectral theorem operator T is unitarily equivalent to M_z , multiplication by the independent variable z , in a direct sum of Hilbert spaces

$$\mathcal{H} = \oplus \int \mathcal{H}(z) d\mu(z)$$

where μ is a scalar positive measure on \mathbb{C} . The measure μ is called a spectral measure of T .

If T is a unitary or self-adjoint operator, its spectral measure μ is supported on the unit circle or on the real line, respectively. Via Radon decomposition, μ can be decomposed into

a singular and absolutely continuous parts $\mu = \mu_s + \mu_{ac}$. The singular component μ_s can be further split into singular continuous and pure point parts. For unitary or self-adjoint T we denote by T_{ac} the restriction of T to its absolutely continuous part, i.e. T_{ac} is unitarily equivalent to

$$M_t|_{\oplus \int \mathcal{H}(t) d\mu_{ac}(t)}.$$

Similarly, define the singular, singular continuous and the pure point parts of T , denoted by T_s , T_{sc} and T_{pp} , respectively.

In terms of the spectral representation described above, the property of cyclicity, as defined in the introduction, is equivalent to the property that all Hilbert spaces $\mathcal{H}(t)$ are one-dimensional and the space

$$\oplus \int \mathcal{H}(t) d\mu(t)$$

can be identified with $L^2(\mu)$. Cyclic vectors for T correspond to functions with full support in $L^2(\mu)$, i.e. those functions that are non-zero almost everywhere with respect to μ .

2.2. Self-adjoint rank-one perturbations. Let T be a normal operator on a Hilbert space \mathcal{H} and let $\varphi \in \mathcal{H}$ be a non-zero vector. An alternative definition of the spectral measure of T can be given as follows. Notice that there exists a unique measure μ on \mathbb{C} such that

$$((T - \lambda \mathbf{I})^{-1} \varphi, \varphi)_{\mathcal{H}} = \int_{\mathbb{R}} \frac{d\mu(t)}{t - \lambda},$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If μ is such a measure we say that μ is the spectral measure of T with respect to the vector φ . Note that such a measure is unique, once T and φ are fixed. The operator T is bounded if and only if μ is compactly supported.

Let A be a self-adjoint operator and let φ be its cyclic vector. Consider the family of rank-one perturbations

$$(2.1) \quad A_\alpha = A + \alpha(\cdot, \varphi)\varphi, \quad \text{for } \alpha \in \mathbb{R}.$$

It is not difficult to show that then φ will be a cyclic vector for A_α for all $\alpha \in \mathbb{R}$. Denote by μ_α the spectral measure of A_α with respect to φ . In these notations $\mu = \mu_0$.

In virtue of the spectral theorem, one can always assume that $\mathcal{H} = L^2(\mu)$, $A = M_t$ and $\varphi = \mathbf{1} \in L^2(\mu)$. Denote by V_α the operator of spectral representation for A_α , i.e. the unitary operator $V_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$ such that $V_\alpha A_\alpha = M_t V_\alpha$ and $V_\alpha \varphi = \mathbf{1}$. An explicit formula for V_α was recently derived in [15].

For unbounded A (i.e. not compactly supported μ), we always assume that the spectral measure μ corresponding to φ satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1 + |t|} < \infty.$$

Using the standard terminology, this means that we consider the class of singular form bounded perturbations and assume that $\varphi \in \mathcal{H}_{-1}(A) \supset \mathcal{H}$, i.e. that $A\varphi \in \mathcal{H}$. Notice that if $\varphi \notin \mathcal{H}_{-1}(A)$ then the formal expression (2.1) does not possess a unique self-adjoint extension, see for instance [14].

The Aronszajn–Donoghue theory analyzes the spectrum of the perturbed operator under rank-one perturbations. We will use the following well-known statement:

Theorem 2.1 ([22]). *For non-equal coupling constants $\alpha \neq \beta$, the singular parts $(\mu_\alpha)_s$ and $(\mu_\beta)_s$ are mutually singular.*

2.3. Aleksandrov–Clark theory and unitary rank-one perturbations. By H^p we will denote the standard Hardy spaces in the unit disk. Recall that a function $\theta \in H^\infty$ is called inner, if $|\theta(z)| = 1$ for almost every $|z| = 1$. The (scalar valued) model space K_θ is defined as $K_\theta = H^2 \ominus \theta H^2$. Such spaces play an important role in complex function theory and functional analysis, see for instance [17].

The model operator on a space K_θ is defined as $S_\theta = P_\theta S$, where P_θ denotes the orthogonal projection onto K_θ , while S is the shift operator given by $Sf(z) = zf(z)$ for $f \in H^2$. The adjoint to the shift operator is the so-called backward shift operator defined as

$$S^*f = \frac{f(z) - f(0)}{z}.$$

Let θ be an inner function. To simplify the formulas we will assume that $\theta(0) = 0$. In [3] Clark showed that the family of rank-one perturbations

$$(2.2) \quad \tilde{U}_\gamma = S_\theta + \gamma(\cdot, S^*\theta) \quad \text{for } \gamma \in \mathbb{T}$$

consists of unitary operators on K_θ , and - vice versa - that all unitary rank-one perturbations of the model operator $S_\theta = P_\theta S|_{K_\theta}$ are given by (2.2).

It is well-known that the vector $1 \in K_\theta$ is cyclic for all operators U_γ in the above family. By σ_γ denote the spectral measures of \tilde{U}_γ with respect to the function $\mathbf{1}$, i.e. such measures that the identity

$$((\tilde{U}_\gamma + zI)(\tilde{U}_\gamma - zI)^{-1}\mathbf{1}, \mathbf{1}) = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma_\gamma(\xi)$$

holds true for all $\alpha \in \mathbb{T}$.

One of the main results of the Clark theory says that the spectral measures of U_γ are defined by the identity

$$(2.3) \quad \frac{\theta + \gamma}{\theta - \gamma} = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\sigma_\gamma(\xi).$$

The Clark operator is the unitary operator $\Phi_\gamma : K_\theta \rightarrow L^2(\sigma_\gamma)$ such that $\Phi_\gamma \tilde{U}_\gamma = M_z \Phi_\gamma$, where M_z is the operator that acts as multiplication by the independent variable in $L^2(\sigma_\gamma)$. In other words, the Clark operator is the spectral representation of \tilde{U}_γ .

Notice that the spectral representation of $V_\alpha : L^2(\sigma) \rightarrow L^2(\sigma_\alpha)$ from the previous subsection on self-adjoint rank-one perturbations corresponds to the composition operator $\Phi_\gamma \Phi_1^* : L^2(\sigma_1) \rightarrow L^2(\sigma_\gamma)$ in the case of unitary rank-one perturbations.

The situation becomes more complicated without the assumption that the spectrum is purely singular. In the case of non-trivial absolutely continuous spectrum the model space consists of pairs of functions analytic inside and outside of the unit disk, see [16].

However, many of the formulas of the Aleksandrov–Clark theory remain valid in the non-singular case. If μ is a positive finite measure on the unit circle, Denote by U_1 the operator of multiplication by z in $L^2(\mu)$. Let θ be a bounded holomorphic function in the unit disk \mathbb{D} that satisfies (2.3) for $\gamma = 1$ and $\sigma_1 = \mu$. If μ is not singular, θ is not inner but still belongs to the unit ball of H^∞ , i.e. $|\theta| \leq 1$. Nonetheless, one can still consider a family of measures σ_γ defined by (2.3). This family will consist of spectral measures of unitary rank-one perturbations of U_1 corresponding to the vector $\mathbf{1} \in L^2(\mu)$, with U_γ defined as

$$U_\gamma = U_1 + (\gamma - 1)(\cdot, U_1^* \mathbf{1})\mathbf{1}, \quad \gamma \in \mathbb{T},$$

see [20].

We will also use the following two theorems from the Aleksandrov-Clark theory:

Theorem 2.2 (Aleksandrov's spectral averaging, see e.g. [20]). *For $f \in L^1(\mathbb{T}, dm)$ we have*

$$\int f dm = \int \left(\int f d\sigma_\gamma \right) dm(\gamma).$$

It is well known that the adjoint $\Phi_\gamma^* : L^2(\sigma_\gamma) \rightarrow K_\theta$ of the Clark operator can be represented using the normalized Cauchy transform

$$\Phi_\gamma^* h = \frac{\mathcal{K}_{h\sigma_\gamma}}{\mathcal{K}_{\sigma_\gamma}},$$

where \mathcal{K} stands for the Cauchy transform in \mathbb{D} :

$$\mathcal{K}_{\sigma_\gamma}(z) = \int_{\mathbb{T}} \frac{d\sigma_\gamma(\xi)}{1 - \bar{\xi}z}, \quad \text{and} \quad \mathcal{K}_{h\sigma_\gamma}(z) = \int_{\mathbb{T}} \frac{h(\xi)d\sigma_\gamma(\xi)}{1 - \bar{\xi}z}.$$

Theorem 2.3 ([18]). *For any $f \in L^1(\sigma_\gamma)$,*

$$\lim_{r \rightarrow 1} \frac{\mathcal{K}_{f\sigma_\gamma}(rz)}{\mathcal{K}_{\sigma_\gamma}(rz)} = f(z), \text{ for } (\sigma_\gamma)_s\text{-a.e. } z \in \mathbb{T}.$$

Theorems 2.2 and 2.3, hold true even if the family of spectral measures possesses non-trivial absolutely continuous parts, although the normalized Cauchy transform cannot be interpreted as a vector from K_θ .

Via the standard agreement, every function from a Hardy space H^p is identified with its boundary values on the circle \mathbb{T} . For a function $f \in K_\theta$, denote by \tilde{f} the function $\theta \bar{f}$ on \mathbb{T} . Note that $f \in K_\theta$ and $f(0) = 0$ imply that $\theta \bar{f} \in K_\theta$. A function $f \in K_\theta$ is called a Hermitian element, if $\tilde{f} = f$. Notice that Hermitian functions satisfy $f(0) = 0$.

The following simple statement plays an important role in section 4.

Theorem 2.4 ([18]). *Let $f \in K_\theta$. Then f is a Hermitian element if and only if*

$$(2.4) \quad \arg(\Phi_\gamma f) = \frac{\arg \gamma}{2} (\bmod \pi), \text{ } \sigma_\gamma\text{-a.e.},$$

and $\int f d\sigma_\gamma = 0$ for some $\gamma \in \mathbb{T}$. If f is a Hermitian element then f satisfies (2.4) and $\int f d\sigma_\gamma = 0$ for any $\gamma \in \mathbb{T}$.

Note that spaces K_θ , the Aleksandrov-Clark theory and all its basic results discussed in this section can be equivalently re-stated in the case of the real line (upper half-plane). The Cauchy transforms in \mathbb{D} will have to be replaced with their analogues in \mathbb{C}_+ , see section 2.5 for definitions. Similarly, results on rank-one perturbations of self-adjoint operators can be reformulated for the families of unitary rank-one perturbations and vice-versa, see for instance [19]. In the rest of the paper we will utilize both settings in our statements and proofs.

2.4. Spaces of Paley–Wiener functions. For $a > 0$ the class of Paley–Wiener functions on \mathbb{R} is given by

$$\text{PW}_a = \{\hat{f} : f \in L^2(-a, a)\},$$

where $\hat{f}(z) = \int e^{-izt} f(t) dt$ denotes the classical Fourier transform of f . Alternatively, the Paley–Wiener space can be characterized as the space of entire functions of exponential type at most a whose boundary values on the real line are square summable with respect to Lebesgue measure.

The Paley–Wiener space PW_a is closely related to the model space K_θ for the inner function

$$\theta_a(z) = \theta(z) = e^{-2a \frac{1+z}{1-z}}$$

in the unit disk. To establish the connection, consider the conformal map

$$\psi(z) = \frac{z-i}{z+i}$$

from \mathbb{C}_+ to \mathbb{D} . Denote $\vartheta(z) = \vartheta_a(z) = e^{2iaz}$. Note that

$$\vartheta_a(z) = \theta_a(\psi(z)).$$

By $K_\vartheta^\mathbb{R}$ denote the space obtained from K_θ by composing all functions from K_θ with ψ , i.e.

$$K_\vartheta^\mathbb{R} = \{f(\psi) | f \in K_\theta\}.$$

Then the space

$$e^{-iaz} K_\vartheta^\mathbb{R} = \{e^{-iaz} f(\psi) | f \in K_\theta\}$$

is equal to the space of entire functions of exponential type at most a and with boundary values on \mathbb{R} that are square summable with respect to the measure $(1+x^2)^{-1} dx$.

Hence we have

$$\text{PW}_a \subset e^{-iaz} K_{\vartheta_a}^\mathbb{R} \quad \text{for } 0 < a.$$

Further, one can prove that the codimension is equal to 1 and

$$e^{-iaz} K_{\vartheta_a}^\mathbb{R} \ominus \text{PW}_a$$

consists of constant functions.

2.5. Cauchy transform. If τ is the spectral measure of a self-adjoint operator A corresponding to the vector φ then the Cauchy transform of τ ,

$$(2.5) \quad \mathcal{K}_\tau(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\tau(t)}{t-z}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

is equal to the corresponding resolvent function of A :

$$\mathcal{K}_\tau(z) = ((A - z\mathbf{I})^{-1}\varphi, \varphi) = \int \frac{d\mu(t)}{t-z}.$$

This connection allows one to apply complex analysis in spectral problems.

We use the notation

$$(\mathcal{K}_\tau)_+(x) = \lim_{y \rightarrow 0} \mathcal{K}_\tau(x + iy) \quad \text{and} \quad (\mathcal{K}_\tau)_-(x) = \lim_{y \rightarrow 0} \mathcal{K}_\tau(x - iy)$$

for $x \in \mathbb{R}$. By a theorem of Privalov,

$$(2.6) \quad (\mathcal{K}_\tau)_-(x) - (\mathcal{K}_\tau)_+(x) = 2\pi i \frac{d\tau}{dx}(x)$$

for Lebesgue-a.e. $x \in \mathbb{R}$.

3. ARBITRARY NON-ZERO VECTORS YIELD CYCLIC VECTORS FOR ALMOST ALL PARAMETERS

Let A be a self-adjoint (possibly unbounded) operator on a separable Hilbert space \mathcal{H} and let φ be a cyclic vector for A . Define the family of self-adjoint rank-one perturbations of A , A_α as in subsection 2.2. Recall that $(A_\alpha)_{ac}$ and $(A_\alpha)_s$ denote the absolutely continuous part and the singular part of the operator A_α , respectively.

Further notice that $(A_\alpha)_s \oplus (A_\alpha)_{ac}$, since $(\mu_\alpha)_s \perp (\mu_\alpha)_{ac}$.

Theorem 3.1. *Let A_α be a family of self-adjoint rank-one perturbations in a Hilbert space \mathcal{H} given by (2.1). Let $0 \neq f \in \mathcal{H}$. Then*

- 1) *The function f is a cyclic vector for $(A_\alpha)_{ac}$ for all but a countable number of $\alpha \in \mathbb{R}$.*
- 2) *The function f is a cyclic vector for $(A_\alpha)_s$ for Lebesgue a.e. $\alpha \in \mathbb{R}$.*

In the following subsection we prove an equivalent reformulation of this theorem in terms of its spectral representation.

3.1. Proof of Theorem 3.1. As we mentioned before, in view of the spectral theorem, instead of dealing with a general family of self-adjoint rank-one perturbations given by equation (2.1) one can consider the self-adjoint rank-one perturbations $A_\alpha = M_t + \alpha(\cdot, \mathbf{1})_{L^2(\mu)} \mathbf{1}$ on $L^2(\mu)$. Let the spectral operator $V_\alpha : L^2(\mu) \rightarrow L^2(\mu_\alpha)$ be as defined in subsection 2.2. In these settings theorem 3.1 can be stated as follows.

Theorem 3.2. *Let $0 \neq f \in L^2(\mu)$. Then*

- 1) *The function $f_\alpha = V_\alpha f \in L^2(\mu_\alpha)$ is not equal to zero $(\mu_\alpha)_{ac}$ -a.e. for all but a countable number of $\alpha \in \mathbb{R}$.*
- 2) *The function $f_\alpha = V_\alpha f \in L^2(\mu_\alpha)$ is not equal to zero $(\mu_\alpha)_s$ -a.e. for Lebesgue a.e. $\alpha \in \mathbb{R}$.*

Remarks. (a) Obviously one cannot expect to obtain the above conclusion of cyclicity for all $\alpha \in \mathbb{R}$. It is always possible to start with f that is zero on a set of positive μ_0 -measure that is, therefore, not cyclic for $\alpha = 0$. In fact, under the conditions of Theorem 3.2, we cannot replace "countable" with "finite."

(b) In the case of purely singular spectral measures for some natural classes of f the conclusion can be strengthened to "all but one" α , see section 4 below.

Our next example says that, in general, if f is not Hermitian, then f can be non-cyclic for uncountably many corresponding rank-one perturbations. In particular, in the conclusion of Theorem 3.2 the distinction between the singular and the absolutely continuous is necessary. As was mentioned above, throughout the rest of the paper we will switch between self-adjoint and unitary settings as a matter of convenience. An analogous discussion can always be carried out in the other case.

Example 1. Consider the setting of rank-one unitary perturbation described in subsection 2.3. We will construct a bounded holomorphic function θ such that for the family of spectral measures $(\sigma_\gamma)_{\gamma \in \mathbb{T}}$ defined by (2.3) and the corresponding family of unitary operators U_γ there exists a non-zero function $f \in L^2(\sigma_1)$ such that $f_\gamma = \Phi_\gamma \Phi_1^* f$ is non-cyclic in $L^2(\sigma_\gamma)$ for an uncountable set of γ 's. Note that in this example θ is not inner and the corresponding operators have nontrivial absolutely continuous parts.

Let C be a Cantor (closed uncountable) subset of the unit circle \mathbb{T} . Let w be the continuous function on \mathbb{T} defined by $w(\xi) = \text{dist}^2(\xi, C)$. Denote by $(\sigma_\gamma)_{\gamma \in \mathbb{T}}$ the system of probability measures which is the family of Clark measures for some inner function θ , and such that the measure σ_1 coincides, up to a multiplicative constant, with the measure $w dm$, where m is the Lebesgue measure on \mathbb{T} . Note that, by definition of σ_1 , we have

$$\int \frac{1}{|x - y|^2} d\sigma_1(y) < \infty$$

for any $x \in C$. It is well known (see, e.g. [4]) that the last condition implies that each point of C is a point mass for one of the measures σ_γ . Since C is uncountable, we conclude that uncountably many σ_γ 's have atoms on the set C .

Let now F be an outer function with modulus equal to w almost everywhere on \mathbb{T} . Consider $f = F/w$. Then f is a unimodular function on \mathbb{T} , and we have $f\sigma_1 = fw = F$, hence $\mathcal{K}_{f\sigma_1} = 0$ on the set C . So we have

$$\frac{\mathcal{K}_{f\sigma_1}}{\mathcal{K}_{\sigma_1}} = 0 \quad \text{on } C.$$

In virtue of Lemma 3.3, it follows that $\Phi_\gamma \Phi_1^* f = 0$ on the (uncountable) set of those $\gamma \in \mathbb{T}$ for which σ_γ has a point mass on C . Hence f is not cyclic for uncountably many operators U_γ .

Proof of part 1) of Theorem 3.2. Define the set

$$\Sigma_\alpha = \{x \in \text{supp}(\mu_\alpha)_{\text{ac}} : f_\alpha(x) = 0\}.$$

The goal is to show that $(\mu_\alpha)_{\text{ac}}(\Sigma_\alpha) = 0$ for all but a countable number of parameters α .

Assume that f_α is not cyclic for uncountably many $\alpha \in \mathbb{R}$, i.e. assume that for some $S \subset \mathbb{R}$, S uncountable, we have $(\mu_\alpha)_{\text{ac}}(\Sigma_\alpha) > 0$ for all $\alpha \in S$. Then $|\Sigma_\alpha| > 0$ for all $\alpha \in S$. Since S is uncountable

$$(3.1) \quad |\Sigma_\alpha \cap \Sigma_\beta| > 0 \quad \text{for some } \alpha, \beta \in S \text{ with } \alpha \neq \beta.$$

Let us fix α and β satisfying (3.1) and investigate the jump behavior in the first equation of (3.3) below. By Fatou's jump theorem, see equation (2.6), we have

$$(\mathcal{K}_{f_\alpha \mu_\alpha})_-(x) - (\mathcal{K}_{f_\alpha \mu_\alpha})_+(x) = 2\pi i \frac{d(f_\alpha \mu_\alpha)}{dx}(x) = 0 \quad \text{Lebesgue-a.e. } x \in (\Sigma_\alpha \cap \Sigma_\beta),$$

because $f_\alpha = 0$ on Σ_α .

Similarly $\mathcal{K}_{f_\beta \mu_\beta}$ has no jump Lebesgue almost everywhere on $\Sigma_\alpha \cap \Sigma_\beta$. On the other hand \mathcal{K}_{μ_β} has a non-zero jump Lebesgue almost everywhere on $(\Sigma_\alpha \cap \Sigma_\beta) \subset \text{supp}(\mu_\alpha)_{\text{ac}}$.

Hence, while the right hand side in the first equation of (3.3) has a jump, the left hand side does not jump Lebesgue-a.e. on $\Sigma_\alpha \cap \Sigma_\beta$ where we have (3.1), and we arrive at a contradiction. Therefore the assumption that f_α is not cyclic for uncountably many $\alpha \in \mathbb{R}$ cannot be maintained. \square

Proof of part 2) of Theorem 3.2. Denote by S_α the essential support of the measure $(\mu_\alpha)_s$ defined as the set of points where the Radon derivative of $(\mu_\alpha)_s$ is infinite. It follows from the the Aronszajn–Donoghue theorem, Theorem 2.1, that the sets S_α are disjoint. Define the set

$$\Omega_\alpha = \{x \in S_\alpha : f_\alpha(x) = 0\}.$$

The goal is to show that $(\mu_\alpha)_s(\Omega_\alpha) = 0$ for Lebesgue a.e. $\alpha \in \mathbb{T}$.

Assume that f_α is not cyclic for a set of α 's with positive Lebesgue measure, i.e. assume that for some $S \subset \mathbb{R}$, $|S| > 0$ we have

$$\mu_\alpha(\Omega_\alpha) > 0 \quad \text{for all } \alpha \in S.$$

In virtue of Theorem 2.3, (or, more precisely, its analog for the real line) we have for the boundary values

$$(3.2) \quad \frac{\mathcal{K}_{f_\alpha \mu_\alpha}(x + iy)}{\mathcal{K}_{\mu_\alpha}(x + iy)} \xrightarrow{y \rightarrow 0} 0$$

for $(\mu_\alpha)_s$ -a.e. $x \in \Omega_\alpha$ and all $\alpha \in S$. Therefore, there exists a set $M \subset \mathbb{R}$, $|M| > 0$ such that for all $x \in M$ there exists an α such that (3.2) is satisfied.

Using Lemma 3.3 (below), the analytic function

$$\frac{\mathcal{K}_{f_\alpha \mu_\alpha}}{\mathcal{K}_{\mu_\alpha}} = \frac{\mathcal{K}_{f_0 \mu_0}}{\mathcal{K}_{\mu_0}}$$

has zero boundary values on M , a set of Lebesgue measure greater than zero.

Hence

$$\frac{\mathcal{K}_{f_0 \mu_0}}{\mathcal{K}_{\mu_0}} \equiv 0$$

and we must have $f_0 \equiv 0$. But this contradicts the hypothesis that f is a non-zero vector. \square

Let us prove the lemma that was used in the above proofs. Recall the definition (2.5) of the Cauchy transform.

Lemma 3.3 (Aronszajn–Krein-type formula). *Under the hypotheses of Theorem 3.2 we have*

$$(3.3) \quad \mathcal{K}_{f_\alpha \mu_\alpha} = \frac{\mathcal{K}_{f_\beta \mu_\beta}}{1 + (\alpha - \beta)\mathcal{K}_{\mu_\beta}} \quad \text{and} \quad \frac{\mathcal{K}_{f_\alpha \mu_\alpha}}{\mathcal{K}_{\mu_\alpha}} = \frac{\mathcal{K}_{f_0 \mu_0}}{\mathcal{K}_{\mu_0}}.$$

Proof of Lemma 3.3. First consider the case where $\mathbf{1} \in \mathcal{H} = L^2(\mu)$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Combining the second resolvent equation and the fact that $A_\alpha = A_\beta + (\alpha - \beta)(\cdot, \mathbf{1})\mathbf{1}$ we obtain

$$(A_\beta - z\mathbf{I})^{-1} - (A_\alpha - z\mathbf{I})^{-1} = (\alpha - \beta)((A_\alpha - z\mathbf{I})^{-1} \cdot, \mathbf{1})(A_\beta - z\mathbf{I})^{-1} \mathbf{1}.$$

Application to a vector $f \in \mathcal{H}$ and pairing with $\mathbf{1}$ yields

$$((A_\beta - z\mathbf{I})^{-1} f, \mathbf{1}) - ((A_\alpha - z\mathbf{I})^{-1} f, \mathbf{1}) = (\alpha - \beta)((A_\alpha - z\mathbf{I})^{-1} f, \mathbf{1})((A_\beta - z\mathbf{I})^{-1} \mathbf{1}, \mathbf{1}).$$

Recall that $V_\alpha A_\alpha = M_t V_\alpha$, $V_\alpha \mathbf{1} = \mathbf{1}$ and $V_\alpha f = f_\alpha$. With this we obtain

$$\mathcal{K}_{f_\beta \mu_\beta} - \mathcal{K}_{f_\alpha \mu_\alpha} = (\alpha - \beta)\mathcal{K}_{f_\alpha \mu_\alpha} \mathcal{K}_{\mu_\beta},$$

or equivalently,

$$(3.4) \quad \mathcal{K}_{f_\alpha \mu_\alpha} = \frac{\mathcal{K}_{f_\beta \mu_\beta}}{1 + (\alpha - \beta)\mathcal{K}_{\mu_\beta}} = \frac{\mathcal{K}_{f_\beta \mu_\beta}}{\mathcal{K}_{\mu_\beta}} \frac{\mathcal{K}_{\mu_\beta}}{1 + (\alpha - \beta)\mathcal{K}_{\mu_\beta}} = \frac{\mathcal{K}_{f_\beta \mu_\beta}}{\mathcal{K}_{\mu_\beta}} \mathcal{K}_{\mu_\alpha}.$$

In the last equality we used the well-known Aronszajn–Krein formula, which can be obtained from the first equality of (3.4) by using $f = \varphi$ (or equivalently $f_\alpha = \mathbf{1}$).

To obtain the second formula of (3.3), we divide both sides by \mathcal{K}_{μ_α} .

If $\mathbf{1} \in \mathcal{H}_{-1}(A) \setminus \mathcal{H}$ the resolvent formula is slightly more complicated

$$(A_\alpha - \lambda \mathbf{I})^{-1} f = (A_\beta - \lambda \mathbf{I})^{-1} f - \frac{(\alpha - \beta) ((A_\beta - \lambda \mathbf{I})^{-1} f, \mathbf{1})}{1 + (\alpha - \beta) ((A_\beta - \lambda \mathbf{I})^{-1} \mathbf{1}, \mathbf{1})} (A_\beta - \lambda \mathbf{I})^{-1} \mathbf{1}$$

for $f \in \mathcal{H}_{-1}(A)$, see e.g. [12]. When paired with the vector $\mathbf{1}$ this yields

$$\mathcal{K}_{f_\alpha \mu_\alpha} = \left(1 - \frac{(\alpha - \beta) \mathcal{K}_{\mu_\beta}}{1 + (\alpha - \beta) \mathcal{K}_{\mu_\beta}} \right) \mathcal{K}_{f_\beta \mu_\beta} = \frac{\mathcal{K}_{f_\beta \mu_\beta}}{1 + (\alpha - \beta) \mathcal{K}_{\mu_\beta}}.$$

The remainder of the proof for $\mathbf{1} \in \mathcal{H}_{-1}(A) \setminus \mathcal{H}$ now follows similarly to the case of regular perturbations $\mathbf{1} \in \mathcal{H}$. \square

3.2. A Corollary of Lemma 3.3. Let us mention another consequence of Lemma 3.3, although we will not use this fact later in this paper.

Corollary 3.4. *We have*

$$\frac{\mathcal{K}(f_\alpha \mu_\alpha)}{\mathcal{K}(f \mu)}(z) = \frac{1 - \theta(z)}{1 - \bar{\alpha} \theta(z)} \quad m - a.e. \ z \in \mathbb{C} \setminus \mathbb{D}.$$

In particular, the function

$$(3.5) \quad \frac{\mathcal{K}(f_\alpha \mu_\alpha)}{\mathcal{K}(f \mu)}$$

is independent of the choice of $f \in L^2(\mu)$.

Remark. It was R. G. Douglas who observed the independence of the expression (3.5) from the choice of $f \in L^2(\mu)$.

Proof. In order to see the second statement notice that by Lemma 3.3, the expression (3.5) is independent from the choice of $f \in L^2(\mu)$, i.e.

$$\frac{\mathcal{K}(f_\alpha \mu_\alpha)}{\mathcal{K}(f \mu)} = \frac{\mathcal{K}(g_\alpha \mu_\alpha)}{\mathcal{K}(g \mu)} \quad \text{for all } f, g \in L^2(\mu).$$

We obtain the first statement by expanding

$$\frac{\mathcal{K}(f_\alpha \mu_\alpha)}{\mathcal{K}(f \mu)} = \frac{\frac{\mathcal{K}(f_\alpha \mu_\alpha)}{\mathcal{K}_{\mu_\alpha}} \mathcal{K}_{\mu_\alpha}}{\frac{\mathcal{K}(f \mu)}{\mathcal{K}_\mu} \mathcal{K}_\mu}$$

and use Lemma 3.3 for $g \equiv 1$ to cancel the fractions in the numerator and denominator. Further apply equation

$$\mathcal{K}_{\mu_\alpha}(z) = \frac{1}{1 - \bar{\alpha} \theta(z)}$$

(confer of [4]) to the remaining \mathcal{K}_{μ_α} in the numerator as well as \mathcal{K}_μ in the denominator. \square

3.3. Anderson-type Hamiltonians. The following operator is a generalization of most Anderson models discussed in literature.

For $n = 1, 2, \dots$ consider the probability space $\Omega_n = (\mathbb{R}, \mathcal{B}, \mu_n)$, where \mathcal{B} is the Borel sigma-algebra on \mathbb{R} and μ_n is a Borel probability measure. Let $\Omega = \prod_{n=1}^{\infty} \Omega_n$ be a product space with the probability measure \mathbb{P} on Ω introduced as the product measure of the corresponding measures on Ω_n on the product sigma-algebra \mathcal{A} . The elements of Ω are points in \mathbb{R}^∞ , $\omega = (\omega_1, \omega_2, \dots), \omega_n \in \Omega_n$.

Let \mathcal{H} be a separable Hilbert space and let $\varphi_1, \varphi_2, \dots$ be a countable collection of unit vectors in \mathcal{H} . For each $\omega \in \Omega$ define an Anderson-type Hamiltonian on \mathcal{H} as a self-adjoint operator formally given by

$$(3.6) \quad H_\omega = H + V_\omega, \quad V_\omega = \sum_n \omega_n(\cdot, \varphi_n) \varphi_n.$$

Except for degenerate cases, the perturbation V_ω is almost-surely a non-compact operator. It is hence not possible to apply results from classical perturbation theory to study the spectra of H_ω , see e.g. [2] and [10].

In the case of an orthogonal sequence $\{\varphi_n\}$, this operator was studied in [8] and [9].

Probably the most important special case of an Anderson-type Hamiltonian is the discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$

$$Hf(x) = -\Delta f(x) = -\sum_{|n|=1} (f(x+n) - f(x)), \quad \varphi_n(x) = \delta_n(x) = \begin{cases} 1 & x = n \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases}$$

3.4. An application of Theorem 3.1 to Anderson-type Hamiltonians. Let H_ω be the Anderson-type Hamiltonian introduced in equation (3.6). Fix $\omega_0 \in \Omega$. Assume $\varphi \in \mathcal{H}_{-1}(H_{\omega_0})$ is a cyclic vector for H_{ω_0} . Consider operators $H_{\omega_0} + \alpha(\cdot, \varphi)\varphi$, $\alpha \in \mathbb{R}$.

Then (by Theorem 3.1) any non-zero $f \in \mathcal{H}$ is cyclic for $H_{\omega_0} + \alpha(\cdot, \varphi)\varphi$ for almost all $\alpha \in \mathbb{R}$. In particular, for Lebesgue almost every α , the operators $H_{\omega_0} + \alpha(\cdot, \varphi)\varphi$ are cyclic.

In the case where $\varphi \in \text{span}\{\varphi_n\}$, $\varphi = \sum a_n \varphi_n$, we say that φ corresponds to the (possibly non-unique) sequence $\mathbf{a} = (a_1, a_2, \dots)$. Further, the operators $H_{\omega_0} + \alpha(\cdot, \varphi)\varphi$ correspond to ω belonging to the one dimensional affine subspace

$$l(\omega_0, \mathbf{a}) = \{\omega_0 + \alpha(a_1, a_2, a_3, \dots) \mid \alpha \in \mathbb{R}\}.$$

Cyclicity of the operators for almost every ω in any one-dimensional affine subspace is a stronger statement than \mathbb{P} -almost-sure cyclicity that can be found in the literature for some particular cases of our model. In terms of almost-sure cyclicity we obtain the following result.

If $l(\omega_0, \mathbf{a})$ is a one-dimensional affine subspace of \mathbb{R}^∞ , one can introduce Lebesgue measure on l as

$$m(S) = |\{\alpha \mid \omega_0 + \alpha(a_1, a_2, a_3, \dots) \in S\}|$$

for any Borel subset S of l .

A sequence $\{\varphi_n\} \subset \mathcal{H}$ is called a *representing system*, if every vector $\varphi \in \mathcal{H}$ can be represented as a series

$$\varphi = \sum a_n \varphi_n$$

that converges with respect to the norm of \mathcal{H} . Note that, unlike the case of a basis, such a representation does not have to be unique.

Corollary 3.5. *Suppose that $\sum a_n \varphi_n$, $\mathbf{a} = (a_1, a_2, \dots)$, is cyclic for H_{ω_0} . Consider a one-dimensional affine subspace of \mathbb{R}^∞ , $l = l(\omega_0, \mathbf{a})$. Then any non-zero vector φ is cyclic for all $(H_\omega)_{ac}$, $\omega \in l$, except possibly countably many ω , and cyclic for almost every $(H_\omega)_s$, $\omega \in l$, with respect to Lebesgue measure on l .*

In particular, suppose that the probability measure \mathbb{P} is a product of absolutely continuous measures and $\{\varphi_n\}$ is a representing system in \mathcal{H} . Assume that there exists a vector $\psi \in \mathcal{H}$ that is cyclic for H_ω , \mathbb{P} -almost surely. Then any non-zero $\varphi \in \mathcal{H}$ is cyclic for H_ω , \mathbb{P} -almost surely.

It is well-known that if an Anderson-type Hamiltonian is singular almost-surely then it is cyclic almost-surely. The proof of almost-sure cyclicity of the singular part $(H_\omega)_s$ and almost-sure cyclicity of certain specific vectors can be found in [9] and for the discrete Schrödinger operator in [21]. The second part of Corollary 3.5 extends these results showing that if there exists an almost sure cyclic vector then any non-zero vector possesses that property.

Proof of Corollary 3.5. The first statement follows immediately from Theorem 3.1.

Let ω be such that ψ is a cyclic vector for H_ω . Let $\mathbf{a} = (a_1, a_2, \dots) \in \mathbb{R}^\infty$ be the sequence of coefficients such that $\psi = \sum a_n \phi_n$. Then every $0 \neq \varphi \in \mathcal{H}$ is cyclic for a.e. point in $l(\omega, \mathbf{a})$. Since the union of such subspaces covers \mathbb{P} -almost all points of \mathbb{R}^∞ , we obtain the statement. \square

4. GENERAL HERMITIAN ELEMENTS AND RANK-ONE PERTURBATIONS

Let U be a unitary operator. For a vector φ consider the space X defined as the closure of the set of real finite linear combinations of elements of the form

$$(U + U^*)^n \varphi \quad \text{and} \quad \frac{1}{i}(U - U^*)^n \varphi \quad \text{for} \quad n \in \mathbb{Z}.$$

Then a vector $f \in \mathcal{H}$ is *Hermitian* with respect to U and the vector φ , if $f \in X$ and $f \perp \varphi$.

An analogous definition can be given for self-adjoint operators. For a bounded self-adjoint operator A on a separable Hilbert space \mathcal{H} and a vector φ , let $(\text{Re } A)\varphi$ denote the closure of the space of linear combinations of $A^n \varphi$, $n \in \mathbb{N}$ with real coefficients. We say that a vector $f \in \mathcal{H}$ is *Hermitian* with respect to the operator A and the vector φ , if $f \in (\text{Re } A)\varphi$ and $f \perp \varphi$. For general (unbounded) operators $(\text{Re } A)\varphi$ can be defined as the closed span of

$$((A - zI)^{-1} + (A - \bar{z}I)^{-1}) \varphi, \quad z \in \mathbb{C}_+.$$

Note that in the settings of section 2.3, the space X defined above is the set of Hermitian functions from K_θ defined there, see the proof of Theorem 4.1 below.

Let U be a unitary operator on a separable Hilbert space \mathcal{H} . Consider the family

$$(4.1) \quad U_\gamma = U + (\gamma - 1)(\cdot, U^{-1}b)_\mathcal{H} b$$

of rank-one perturbations, $\gamma \in \mathbb{T}$, $b \in \mathcal{H}$ with $\|b\|_\mathcal{H} = 1$. It is well known that U_γ is unitary for all $\gamma \in \mathbb{T}$. Clearly we have $U = U_1$. Without loss of generality, assume that b is cyclic for U , i.e.

$$\text{clos span}\{U^k b : k \in \mathbb{Z}\} = \mathcal{H}.$$

Theorem 4.1. *Consider the family U_γ of rank-one unitary perturbations given by (4.1). Assume that U_γ has purely singular spectrum for some (all) $\gamma \in \mathbb{T}$. Let $0 \neq f \in \mathcal{H}$ be Hermitian with respect to $U = U_1$ and b . Fix a constant $c \in \mathbb{C} \setminus \{0\}$. Then the function $f - cb$ is cyclic for U_γ for all $\gamma \in \mathbb{T} \setminus \{e^{2i \arg c}\}$.*

4.1. Proof of Theorem 4.1. Via the spectral theorem, without loss of generality one can assume that $U = U_1$ is an operator of multiplication by z in $L^2(\sigma_1)$, σ_1 is the spectral measure of U corresponding to b and $b = \mathbf{1} \in L^2(\sigma_1)$. Define the inner function θ so that σ_1 is its Clark measure. Let K_θ , γ , σ_γ as well as Φ_γ and Φ_γ^* be as defined in subsection 2.3. Consider operators \tilde{U}_γ defined in (2.2). Then $U_\gamma = \tilde{U}_\gamma$.

Recall that $f \in K_\theta$ is called Hermitian if $f = \tilde{f}$, where $\tilde{f} = \theta \bar{f}$.

Using Theorem 2.4 one can show that this definition is equivalent to that of a Hermitian element with respect to operator $U = \tilde{U}_1$ and vector $b = S^* \theta$. The condition $f(0) = 0$ (for $f \in K_\theta$) translates into $f \perp b$.

Recall that by Theorem 2.3, the non-tangential limit of $f \in K_\theta$ exist σ_γ -a.e. for all $\gamma \in \mathbb{T}$. Let us denote this non-tangential limit by

$$f_\gamma(z) = \lim_{\xi \rightarrow z} f(\xi) \quad \sigma_\gamma\text{-a.e.}$$

In fact, we can identify these boundary values with one function, say f (slightly abusing notation), on the circle which is defined σ_γ -a.e. for all γ . Indeed, we restricted ourselves to purely singular measures and by the Aronszajn–Donoghue theorem, Theorem 2.1, we have $\sigma_\gamma \perp \sigma_\eta$ for $\gamma \neq \eta$. Our statement now follows from theorem 4.2 below.

Theorem 4.2. *Let $0 \neq f \in K_\theta$ be a Hermitian function and fix any constant $c \in \mathbb{C} \setminus \{0\}$. Then the level sets*

$$\{z \in \mathbb{T} : f(z) = c\}$$

have zero σ_γ -measure for all $\gamma \in \mathbb{T} \setminus \{e^{2i \arg c}\}$. In particular, the function $f - c$ is cyclic for \tilde{U}_γ for all $\gamma \in \mathbb{T} \setminus \{e^{2i \arg c}\}$.

Proof of Theorem 4.2. Pick f and c according to the hypotheses of the theorem. By Theorem 2.4 we have

$$\arg f = \frac{\arg \gamma}{2} \pmod{\pi}$$

with respect to σ_γ almost everywhere.

If γ is such that $f = c$ on a set $S \subset \mathbb{T}$ with $\sigma_\gamma(S) > 0$, then we have

$$\frac{\arg \gamma}{2} \pmod{\pi} = \arg c$$

and therefore $\gamma = e^{2i \arg c}$. □

Remark. In the statement of Theorem 4.2, and therefore Theorem 4.1, the constant c cannot be equal to 0. Indeed, consider K_{z^n} that consists of all polynomials of degree less than n . Let $\beta_1, \dots, \beta_{n-2}$ be points on \mathbb{T} such that $\beta_k^n = \gamma_k$ are different points. Let

$$p(z) = a_{n-1}z^{n-1} + \dots + a_1z$$

be a polynomial with roots at $0, \beta_1, \dots, \beta_{n-2}$. Then

$$\tilde{p}(z) = \bar{a}_1z^{n-1} + \dots + \bar{a}_{n-1}z$$

has roots at the same points. Notice that $p + \tilde{p}$ is a Hermitian element of K_{z^n} whose zero set $Z = \{0, \beta_1, \dots, \beta_{n-2}\}$ satisfies $\sigma_{\gamma_k}(Z) = 1/n > 0$ for $k = 1, 2, \dots, n-2$.

Let us mention the following examples that illustrate theorem 4.2.

Level sets of self-reciprocal polynomials. A polynomial

$$p(z) = \sum_{m=0}^{n-1} a_m z^m$$

of degree less than n is a Hermitian element of K_{z^n} , if and only if

$$(4.2) \quad a_0 = 0 \quad \text{and} \quad a_m = \overline{a_{n-m}}, \quad m = 1, \dots, n-1.$$

Polynomials that satisfy (4.2) are called *self-reciprocal*. Note that the Clark measure σ_γ of $\theta = z^n$ is concentrated on the set of n -th roots of γ .

Hence, if $c \neq 0$ and p is a self-reciprocal polynomial, then by Theorem 4.2 all roots of the equation $p = c$ on \mathbb{T} must be contained in a set of n -th roots of γ for $\gamma \in \mathbb{T}$ given in the statement of the theorem.

Naturally, this simple fact can also be proved directly. If z is such that $p(z) = c$ then

$$c = p(z) = z^n \overline{p(z)} = z^n \bar{c}.$$

Hence $z^n = c/\bar{c}$ where $|c/\bar{c}| = 1$ and $\arg(c/\bar{c}) = 2 \arg c$. The proof of Theorem 4.2 can be viewed as a generalization of this argument.

The non-zero level sets of Paley–Wiener functions. Recall the definition of Paley–Wiener functions from subsection 2.4.

The following statement is an analog to Euler’s Formula $e^{i\theta} = \cos \theta + i \sin \theta$ for Paley–Wiener functions.

Proposition 4.3. *Let $f \in \text{PW}_a$. Then $e^{iaz} f = g_1 + ig_2$ where g_1, g_2 are entire functions such that $g_1, g_2 \in e^{iaz} \text{PW}_a$ and each level set*

$$\{x \in \mathbb{R} \mid g_i(x) = c\}, \quad i = 1, 2; \quad c \neq 0,$$

is contained in the arithmetic progression

$$\left\{ 2 \arg c + \frac{2\pi n}{a} \right\}_{n \in \mathbb{Z}}.$$

Proof. Recall that $K_{\vartheta_a}^{\mathbb{R}}$ was defined as the function space obtained by “mapping” the model space K_θ , where

$$\theta_a(z) = \theta(z) = e^{-2a \frac{1+z}{1-z}}, \quad 0 < a \leq 1$$

from \mathbb{D} to \mathbb{R} using the standard conformal map $\psi : \mathbb{C}_+ \rightarrow \mathbb{D}$, see section 2.4. Then

$$e^{iaz} \text{PW}_a \subset K_{\vartheta_a}^{\mathbb{R}}.$$

Hence we have $e^{iaz} f \in K_{\vartheta}^{\mathbb{R}}$.

Without loss of generality $a = 1$. For the inner function $\theta = e^{-\frac{1+z}{1-z}}$ in \mathbb{D} , the Clark measure $(\sigma_\gamma)_{\gamma \in \mathbb{T}}$ is concentrated on the sequence

$$\psi(\{\arg \gamma + 2\pi n\}) \subset \mathbb{T}.$$

Like in the remark following Theorem 4.2, we can decompose f into $f = g_1 + ig_2$ where the translations \tilde{g}_1 and \tilde{g}_2 of the functions g_1 and g_2 from the upper half plane to the disc are Hermitian in K_θ . By Theorem 4.2

$$\{z \in \mathbb{T} : \tilde{g}_i = c\} \subset \psi(\{\arg \gamma + 2\pi n\}), \quad \arg \gamma = 2 \arg c.$$

Hence the level sets of the functions g_i are contained in arithmetic progressions given in the statement. \square

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